JOURNAL OF APPROXIMATION THEORY 74, 69–82 (1993)

Convex Approximation by Quadratic Splines*

YINGKANG HU

Department of Mathematics and Computer Science, Georgia Southern University, Statesboro, Georgia 30460-8093

Communicated by Dany Leviatan

Received July 10, 1991; accepted in revised form March 31, 1992

Given a convex function f without any smoothness requirements on its derivatives, we estimate its error of approximation by $\mathbf{C}^1$ convex quadratic splines in terms of $\omega_3(f, 1/n)$. © 1993 Academic Press, Inc.

1. INTRODUCTION

Many results regarding the order of a shape preserving spline approximation are known when the function to be approximated has continuous derivative(s). For example, DeVore [4] proves that if $f \in \mathbf{C}^j[0, 1]$, $0 \leqslant j < r$, is monotone, then there exists a monotone spline s of order r with n equally spaced knots such that

$$\|f-s\| \leqslant C_r n^{-j} \omega(f^{(j)}, 1/n), \tag{1.1}$$

where and throughout this paper $\|\cdot\|$ is the uniform norm and ω the usual modulus of continuity. Interested readers can refer to [1–7].

On the other hand, if the function f is merely in $\mathbf{C}[0, 1]$, we know very little. For monotone approximation, DeVore [4] explains why it is desirable to be able to replace the right-hand side of (1.1) by $C_r \omega_r(f, 1/n)$, but as he points out his method of proof only yields $C_r n^{-1} \omega_{r-1}(f', 1/n)$ (when $f \in \mathbf{C}^1[0, 1]$) (see also [7]). He also remarks that one can get ω_r when $r = 1, 2$, but we now know [13] that this is impossible for $r > 3$ so that the most one can get is what DeVore's proof could yield. Here

$$\omega_r(f, t) := \sup_{0 < h \leqslant t} \|\Delta_h^r(f)\|, \qquad t > 0$$

* Research supported in part by the Faculty Research Committee of Georgia Southern University.

0021-9045/93 $5.00

and

$$\Delta_h^r(f, x) = \sum_{i=0}^{r} \binom{r}{i} (-1)^{i+r} f(x+ih), \qquad x, x+rh \in [0, 1].$$

For convex approximation, it is known that a convex function $f \in \mathbf{C}[0, 1]$ can be approximated by (even first degree) convex splines at the rate of $\omega_2(f, 1/n)$. Since $\omega_3(f, 1/n)$ is the optimal order for unconstrained approximation by quadratic splines with n equally spaced knots, we would also like to have the same rate for convex approximation. It should be noted, as Professor X. Yu pointed out to the author, that we cannot replace ω_3 even by $\omega_4(f, 1)$, for if f is a cubic polynomial, then $\omega_4(f, 1) = 0$, and no quadratic spline approximates a cubic polynomial exactly.

It is the purpose of this paper to show that ω_3 can be reached for any continuous function. More precisely, we prove that one can approximate a convex function $f \in \mathbf{C}[0, 1]$ by $\mathbf{C}^1$ convex quadratic splines with at most n knots at the rate of $\omega_3(f, 1/n)$. We recall that a function f defined on an interval I is said to be convex if for any x, $y \in I$ and any $0 \leqslant \lambda \leqslant 1$ we have

$$f(\lambda x + (1-\lambda) y) \leqslant \lambda f(x) + (1-\lambda) f(y). \tag{1.2}$$

If the equality in (1.2) does not hold unless $\lambda = 0$ or 1, then f is said to be strictly convex. Geometrically this means that each point on the chord between $(x, f(x))$ and $(y, f(y))$ is above the graph of f. It is not difficult to see the facts listed below (cf., for example, [10, Sect. 5.5]): Suppose f is convex, then

(1) the tangent line of f at any point $x \in I$, if exists, lies below the graph of f;

(2) on any closed interval contained in I, f is absolutely continuous;

(3) if three points on the graph of f at $x_1 < x_2 < x_3$ are collinear, then the graph between x_1 and x_3 is a line segment.

The following is our main theorem, whose proof will be given in Section 3 at the end of this paper.

THEOREM 1.1. *Let $f \in \mathbf{C}[0, 1]$ be convex, and n a positive integer. Then there is a $\mathbf{C}^1$ convex quadratic spline s with at most n knots in $(0, 1)$, such that*

$$\|f - s\| \leqslant C\omega_3(f, 1/n) \tag{1.3}$$

with C an absolute constant.

In fact, in the proof of Theorem 1.1, we shall construct a spline s with less than $8n$ knots. Since $\omega_3(f, 8/m) \leqslant 8^3 \omega_3(f, 1/m)$ for any $m > 0$, we obtain (1.3) from this. The $8n$ knots are arranged in $2n$ groups which are basically equally spaced. More precisely, we shall first interpolate f by a quadratic polynomial p_i on each subinterval $[x_i, x_{i+2}]$ at x_i, x_{i+1} and x_{i+2}, $i = 0, 1, ..., 2n-2$, where $x_i := i/2n$, $i = 0, 1, ..., 2n$. Since f is convex, so are the resulting overlapping quadratics. We will then transit from p_{i-1} to p_i by a convex spline s_i for each $i = 1, 2, ..., 2n-2$ to get a final global approximating spline. This s_i, which we call the transiting part of the pair, will be defined on some transiting interval $J_i := [t_{i,1}, t_{i,2}]$, and consist of at most two pieces of quadratic polynomials. The final spline s is to be in $\mathbf{C}^1$, this means we should have

$$s_i^{(j)}(t_{i,1}) = p_{i-1}^{(j)}(t_{i,1}), \qquad s_i^{(j)}(t_{i,2}) = p_i^{(j)}(t_{i,2}), \qquad j = 0, 1. \tag{1.4}$$

The first question we face is where to put J_i. It is tempting to put it somewhere inside $[x_i, x_{i+1}]$ to prevent the transiting parts s_i from interfering with each other. It would work perfectly if f were *strictly* convex, but this is not always the case. The possible existence of line segments in the graph of f requires a more sophisticated method. To develop such a method, we need some technical lemmas which are given in the next section.

2. Preliminary Results

When creating a global spline from local polynomials, we will frequently meet the following interpolation problem.

PROBLEM 2.1. *Let $t_1 < t_2$, and let y_1, y_2, s_1 and s_2 be any given real numbers. Find a quadratic spline $s \in \mathbf{C}^1[t_1, t_2]$ such that*

$$s(t_i) = y_i, \qquad s'(t_i) = s_i, \qquad i = 1, 2. \tag{2.1}$$

One answer to this problem is given by the following three simple lemmas by Schumaker [12]. For similar techniques, the reader can also refer to numerous previous papers on shape preserving quadratic spline interpolation by Roulier and various coauthors, as pointed out by a referee of this paper. See [8, 9, 12] and references therein.

LEMMA 2.2. *There is a quadratic polynomial solving Problem* 2.1 *if and only if*

$$\frac{s_1 + s_2}{2} = \frac{y_2 - y_1}{t_2 - t_1}. \tag{2.2}$$

If (2.2) *holds, then the polynomial is given by*

$$s(x) = y_1 + s_1(x - t_1) + \frac{(s_2 - s_1)(x - t_1)^2}{2(t_2 - t_1)}.$$

Remark. Note that $s''(x) \equiv (s_2 - s_1)/(t_2 - t_1) > 0$ whenever $s_2 > s_1$.

LEMMA 2.3. *For every $\xi \in (t_1, t_2)$, there exists a unique quadratic spline s with one simple knot at ξ solving Problem* 2.1, *and it is given by*

$$s(x) = \begin{cases} A_1 + B_1(x - t_1) + C_1(x - t_1)^2, & t_1 \leqslant x < \xi, \\ A_2 + B_2(x - \xi) + C_2(x - \xi)^2, & \xi \leqslant x \leqslant t_2, \end{cases} \tag{2.3}$$

with

$$\begin{aligned} A_1 &= y_1, \qquad B_1 = s_1, \qquad C_1 = (s_\xi - s_1)/2\alpha, \\ A_2 &= A_1 + B_1\alpha + C_1\alpha^2, \qquad B_2 = s_\xi, \qquad C_2 = (s_2 - s_\xi)/2\beta, \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} s_\xi = s'(\xi) &= \frac{2(y_2 - y_1) - (\alpha s_1 + \beta s_2)}{t_2 - t_1}, \\ \alpha = \xi - t_1, &\qquad \beta = t_2 - \xi. \end{aligned} \tag{2.5}$$

LEMMA 2.4. *Let $\delta = (y_2 - y_1)/(t_2 - t_1)$. Then $(s_2 - \delta)(\delta - s_1) \leqslant 0$ implies that s must have an inflection point in the interval $I := [t_1, t_2]$ unless $s_1 = s_2 = \delta$, which is a trivial case. Suppose now that $(s_2 - \delta)(\delta - s_1) > 0$. Then the condition $|s_2 - \delta| < |\delta - s_1|$ implies that for all ξ satisfying*

$$t_1 < \xi \leqslant \bar{\xi} \qquad \textit{with} \quad \bar{\xi} = t_1 + \frac{2(t_2 - t_1)(s_2 - \delta)}{s_2 - s_1}, \tag{2.6}$$

the interpolating spline s in (2.3) *is convex if $s_1 < s_2$, is concave if $s_1 > s_2$. In addition, if $s_1 s_2 \geqslant 0$, then s is also monotone. Similarly, if $|s_2 - \delta| > |\delta - s_1|$, then for all ξ satisfying*

$$\underline{\xi} \leqslant \xi < t_2 \qquad \textit{with} \quad \underline{\xi} = t_2 + \frac{2(t_2 - t_1)(s_1 - \delta)}{s_2 - s_1}, \tag{2.7}$$

s also has the above properties.

Remark. The case of $|s_2 - \delta| = |\delta - s_1|$ is given by Lemma 2.2.

The intervals $(t_1, \bar{\xi}]$ and $[\underline{\xi}, t_2)$ are called the admissible intervals for convexity [5]. To make our further discussion short, from now on the word "parabola" will be used for p_i only if it is true parabola, i.e., $p_i'' > 0$, otherwise the word "line" will be used instead. A close study of our situation with the aid of these three lemmas suggests that although it is always possible to transit from p_{i-1} to p_i inside $[x_i, x_{i+1}]$, there is no way to make s_i convex there if one of p_{i-1} or p_i is a line (Line–Parabola and Parabola–Line cases, or, for short, LP and PL cases), therefore in these cases J_i has to be moved partly outside, across x_i or x_{i+1}. As a consequence, J_i has to be put inside the *open* interval (x_i, x_{i+1}) when they are both parabolas (PP case), to make room for LP and PL transition. More efforts will be made in the proof of Theorem 1.1 to prevent interference of the transiting parts.

We have also to determine the position of the (possible) interior knot ξ_i of s_i to make sure that s_i is indeed convex, and its approximation error is not too large. It is clear that after the position of J_i has been determined, all numbers in (2.1) are fixed, thus s_i depends only on the choice of ξ_i. Because of (1.4), this choice only affects the behavior of s_i in the middle of J_i. This influence can be described in terms of the second derivative s_i'' of s_i as in the following lemma, which shows how s_i'' changes as ξ_i runs over the admissible interval. We make the assumption $\delta = 0$ to simplify the computation, and keep using the notation of Problem 2.1 to avoid complex subscription.

LEMMA 2.5. *Let $I_1 := (t_1, \xi)$ and $I_2 := (\xi, t_2)$. Let $y_1 = y_2$, i.e., $\delta = 0$, and $s_1 < 0 < s_2$. Then the second derivative s'' of s in* (2.3), *which is piecewise constant as a function of x, is a monotone function of ξ. More precisely, we have*:

(1) *if $|s_2| < |s_1|$, then as ξ runs over the admissible interval $(t_1, \bar{\xi}]$, $s''(x)$ runs from $+\infty$ to $(1-(s_1+s_2)/2s_2)(s_2-s_1)/(t_2-t_1)$ for $x \in I_1$, and from $2s_2/(t_2-t_1)$ to 0 for $x \in I_2$*;

(2) *if $|s_2| > |s_1|$, then as ξ runs over the admissible interval $[\underline{\xi}, t_2)$, $s''(x)$ runs from 0 to $-2s_1/(t_2-t_1)$ for $x \in I_1$, and from $(1-(s_1+s_2)/2s_1)$ $(s_2-s_1)/(t_2-t_1)$ to $+\infty$ for $x \in I_2$.*

The results hold true if we simultaneously replace $s_1 < 0 < s_2$ and $+\infty$ by $s_1 > 0 > s_2$ and $-\infty$, respectively.

Proof. We first compute s'' from (2.3), (2.4), and (2.5). For $x \in I_1$,

$$s''(x) = 2C_1 = \frac{s_\xi - s_1}{\xi - t_1} = \frac{(-\alpha s_1 - \beta s_2)/(t_2 - t_1) - s_1}{\xi - t_1}$$

$$= \frac{-(\xi - t_1)s_1 - (t_2 - \xi)s_2 - (t_2 - t_1)s_1}{(t_2 - t_1)(\xi - t_1)}$$

$$= \frac{\xi s_2 - \xi s_1 + 2t_1 s_1 - t_2 s_1 - t_2 s_2}{(t_2 - t_1)(\xi - t_1)}$$

$$= \frac{s_2 - s_1}{t_2 - t_1} - \frac{s_1 + s_2}{\xi - t_1}. \tag{2.8}$$

Similar computation shows that for $x \in I_2$,

$$s''(x) = 2C_2 = \frac{s_2 - s_\xi}{t_2 - \xi} = \frac{s_2 - s_1}{t_2 - t_1} + \frac{s_1 + s_2}{t_2 - \xi}. \tag{2.9}$$

It is now easy to see that s'' is monotone in ξ. For the values of s'', we only show the case of $|s_2| > |s_1|$ and $x \in I_1$, the other cases are similar. Indeed, when

$$\xi = \underline{\xi} = t_2 + \frac{2(t_2 - t_1)s_1}{s_2 - s_1},$$

$$\xi - t_1 = \frac{(t_2 - t_1)(s_1 + s_2)}{s_2 - s_1},$$

therefore

$$s''(x) = \frac{s_2 - s_1}{t_2 t_1} - \frac{(s_1 + s_2)(s_2 - s_1)}{(t_2 - t_1)(s_1 + s_2)} = 0.$$

And as $\xi \to t_2 -$,

$$s''(x) \to \frac{s_2 - s_1}{t_2 - t_1} - \frac{s_2 + s_1}{t_2 - t_1} = \frac{-2s_1}{t_2 - t_1}. \quad \blacksquare$$

Our next lemma says that we can transit from a line to a parabola on an interval $[t_1, t_2]$ which contains at least one of their intersection points, so that the transiting spline is convex and lies in certain region, which enables us to control the approximation error. This will be used for LP (or PL) transition. The interval $[t_1, t_2]$ here corresponds to the transiting interval $J_i = [t_{i,1}, t_{i,2}]$, and the point $(\bar{x}, \bar{y})$ corresponds to the intersection point $(x_i, f(x_i))$ (or $(x_{i+1}, f(x_{i+1}))$) of p_{i-1} and p_i inside J_i.

LEMMA 2.6 (for LP or PL Transition). *Let $\underline{x} < t_1 < \bar{x} < t_2$, and y_2 and $\underline{y}$ be two real numbers. Let p be the quadratic polynomial passing through $(\underline{x}, \underline{y})$ and (t_2, y_2) with $p'' \equiv a > 0$, and l_0 the line segment passing through the same two points. Let l be the straight line passing through $(t_1, y_1) := (t_1, l_0(t_1))$ and $(\bar{x}, \bar{y}) := (\bar{x}, p(\bar{x}))$. Then there exists a $\mathbf{C}^1$ convex quadratic* *spline s on $[t_1, t_2]$ with a possible simple knot $\xi \in (t_1, t_2)$ which transits from l to p in the sense that*

$$s^{(j)}(t_1) = l^{(j)}(t_1), \qquad s^{(j)}(t_2) = p^{(j)}(t_2), \qquad j = 0, 1. \tag{2.10}$$

Moreover, $\max(l, p) \leqslant s \leqslant l_0$ *on* $[t_1, t_2]$.

If we exchange the positions of t_1 and $\bar{x}$, i.e., replace $\underline{x} < t_1 < \bar{x} < t_2$ by $\underline{x} < \bar{x} < t_1 < t_2$, then there exists a convex transiting spline s on $[\underline{x}, t_1]$ with similar properties.

Proof. We can suppose $y_2 = \underline{y} = 0$, the general case can be obtained by linear change of variable. Now l_0 becomes $y = 0$ (see Fig. 1.) We only prove the case $\underline{x} < t_1 < \bar{x} < t_2$, the other case is similar. It is easy to see that $p(x) = \frac{1}{2}a(x - t_2)(x - \underline{x})$ and $l(x) = s_1(x - t_1)$ with $s_1 := \bar{y}/(\bar{x} - t_1)$, that $p < 0$ on $(\underline{x}, t_2)$, in particular, $\bar{y} = p(\bar{x}) < 0$, therefore, $s_1 < 0$. A simple computation shows that $s_2 := p'(t_2) = \frac{1}{2}a(t_2 - \underline{x}) > 0$, $s_4 := p'(\underline{x}) = \frac{1}{2}a(\underline{x} - t_2) = -s_2 < 0$, and

$$p''(x) \equiv a = \frac{s_2 - s_4}{t_2 - \underline{x}} = \frac{2s_2}{t_2 - \underline{x}}. \tag{2.11}$$

The first part of the lemma is now given by Lemmas 2.2 and 2.4. For the second part we need $s \leqslant 0$, $s \geqslant l$ and $s \geqslant p$ on $[t_1, t_2]$.

It is obvious from the properties of convex functions listed in Section 1 that $s \leqslant 0$ for $s(t_1) = s(t_2) = 0$ and s is convex. It is also obvious that $s \geqslant l$ for l is the tangent line of s at t_1. We now show in three cases that we can have $s \geqslant p$.

Case 1. $|s_1| = |s_2|$. We have $s_1 = -s_2 = s_4$. Lemma 2.2 says that s is the unique quadratic polynomial satisfying (2.10) with

$$s'' \equiv \frac{s_2 - s_1}{t_2 - t_1} = \frac{s_2 - s_4}{t_2 - t_1} > \frac{s_2 - s_4}{t_2 - \underline{x}} = p''.$$

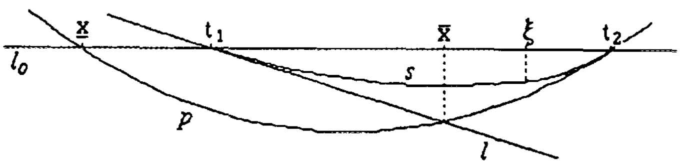

FIGURE 1

By Taylor's Theorem, we have

$$s(x)-p(x)=[s(t_2)-p(t_2)]+[s'(t_2)-p'(t_2)](x-t_2)$$
$$+\int_{t_2}^{x}(x-t)[s''(t)-p''(t)]\,dt$$
$$=\int_{x}^{t_2}(t-x)[s''(t)-p''(t)]\,dt\geqslant 0. \tag{2.12}$$

Case 2. $|s_1|<|s_2|$. Here s is not unique and depends on the position of the knot ξ in the admissible interval $[\xi, t_2)$ for convexity. Lemma 2.5 says that $\lim_{\xi\to t_2-} s''=+\infty$ on $I_2=(\xi, t_2)$, thus we can choose ξ close enough to t_2 so that on I_2 $s''>p''$, hence $s\geqslant p$ by (2.12), in particular, $s(\xi)\geqslant p(\xi)$, (in fact, $s(\xi)>p(\xi)$). If, for this choice of ξ, $s''\geqslant p''$ on $I_1=(t_1,\xi)$, the same argument applies. If $s''<p''$ there, we turn to look at the difference $\phi:=s-p$, which is a parabola open downward with $\phi(t_1)>0$ and $\phi(\xi)>0$, therefore $\phi>0$, i.e., $s>p$, on the whole interval I_1.

Case 3. $|s_1|>|s_2|$. This is similar to Case 2. Here the admissible interval is $(t_1, \bar{\xi}]$. By Lemma 2.5, $\lim_{\xi\to t_1+} s''=2s_2/(t_2-t_1)>2s_2/(t_2-\underline{x})=p''$ on I_2. Thus we can choose ξ close enough to t_1 so that $s''>p''$ on I_2, therefore $s\geqslant p$ on $[t_1, t_2]$ by exactly the same argument as that in Case 2. ∎

Lemma 2.7 below will be used for transition between two parabolas. It says that in this case J_i can be put inside the open interval (x_i, x_{i+1}) to make room for possible neighboring LP (or PL) transition, that there exists a transiting part s_i lying between p_{i-1} and p_i, thus approximating f no worse than they do. Again, we use the notation of (2.1) to avoid complex subscription.

LEMMA 2.7 (for PP Transition). *Let $(\underline{x}, \underline{y})$ and $(\bar{x}, \bar{y})$ be two points in the plane with $\underline{x}<\bar{x}$. Let p_1 and p_2 be two strictly convex quadratic polynomials, both passing through the two points. Then there exists a convex* $\mathbf{C}^1$ *quadratic spline s on an interval $[t_1, t_2]\subset(\underline{x}, \bar{x})$ with a (possible) simple knot $\xi\in(t_1, t_2)$, such that*

$$s^{(j)}(t_1)=p_1^{(j)}(t_1), \qquad s^{(j)}(t_2)=p_2^{(j)}(t_2), \qquad j=0, 1, \tag{2.13}$$

and s lies between p_1 and p_2.

Proof. We again make some simplification, the general case can be obtained by linear change of variable. Suppose $(\underline{x}, \underline{y})=(-1, 0)$ and $(\bar{x}, \bar{y})=(1, 0)$. Then the two polynomials can be written as $p_1=\frac{1}{2}a_1(x^2-1)$ and $p_2=\frac{1}{2}a_2(x^2-1)$, where a_1 and a_2 denote their second derivatives. We also assume $a_1 \neq a_2$, since the result is trivial if $a_1 = a_2$. We can further assume $0 < a_1 < a_2$, since the other case, $0 < a_2 < a_1$, is very similar (in fact, they are geometrically symmetric; see Fig. 2).

We prove this lemma also by Lemmas 2.4 and 2.5. Let us check their conditions first. Since $p_1(\pm 1) = p_2(\pm 1) = 0$, $p_2(0) = -\frac{1}{2}a_2 < p_1(0) = -\frac{1}{2}a_1 < 0$, it is easy to prove that

$$p_2(x) < p_1(x) < 0, \qquad x \in (-1, 1). \tag{2.14}$$

We choose t_1 arbitrarily in $(-1, 0)$, and define t_2 as $\sqrt{1 - (a_1/a_2)(1 - t_1^2)}$, the positive root of the equation $p_2(x) = p_1(t_1)$, then $-1 < t_1 < 0 < t_2 < 1$. Denote

$$y_i := p_i(t_i), \qquad s_i := p_i'(t_i) = a_i t_i, \qquad i = 1, 2,$$

we have $y_1 = y_2$, i.e., $\delta = 0$, and $s_1 < 0 < s_2$. The last thing to check before using the lemmas is the relationship between $|s_1|$ and $|s_2|$. It turns out that $|s_1| < |s_2|$. This is because

$$t_2^2 - t_1^2 = 1 - \frac{a_1}{a_2}(1 - t_1^2) - t_1^2 = \left(1 - \frac{a_1}{a_2}\right)(1 - t_1^2) > 0,$$

thus $t_2 > |t_1|$, and $s_2 = a_2 t_2 > a_1 t_2 > a_1 |t_1| = |s_1|$.

We now apply Lemma 2.4 to the interval $[t_1, t_2]$ and obtain all the results but the position of the resulting convex spline s, which depends on the choice of the knot $\xi \in I_c := [\underline{\xi}, t_2)$. If we can choose ξ so that

$$s'' < a_1 \quad \text{on} \quad I_1 = (t_1, \xi), \qquad s'' > a_2 \quad \text{on} \quad I_2 = (\xi, t_2), \tag{2.15}$$

then the same argument as that in Case 2 in the proof of Lemma 2.6 will give that $s \geqslant p_2$ on $[t_1, t_2]$, and a similar argument will show $p_1 \geqslant s$ on the

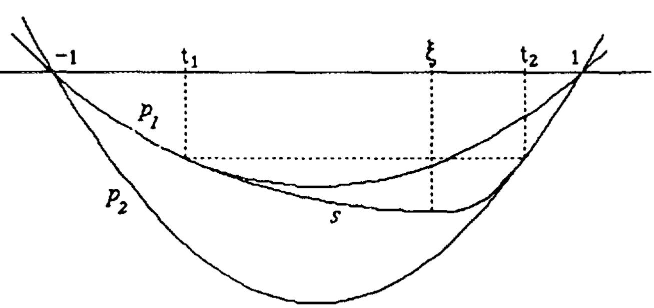

FIGURE 2

same interval, which will finish the proof. This is possible because, by Lemma 2.5,

$$s'' < \frac{-2s_1}{t_2 - t_1} = \frac{-2a_1 t_1}{t_2 - t_1} = \frac{a_1 |t_1| - a_1 t_1}{t_2 - t_1} < \frac{a_1 t_2 - a_1 t_1}{t_2 - t_1} = a_1$$

on I_1 for any $\xi \in I_c$, and $\lim_{\xi \to t_2} s'' = +\infty$ on I_2. If we choose ξ close enough to t_2, then s'' will satisfy both conditions in (2.15). ∎

Remark. Note that only one of the endpoints of the transiting interval $[t_1, t_2]$ can be arbitrarily chosen in one half of the interval $[-1, 1]$, (for example, as the midpoint of the half,) the other will be determined by the lemma and may be very close to 1 or -1, depending on the relative relationship of p_1 and p_2. This makes it more difficult to prevent interference of the transiting parts.

The last lemma we need in the proof of Theorem 1.1 says that our local interpolating polynomials approximate f "nearly" as well as the best ones. Because it is an immediate consequence of Newton's formula for interpolating polynomials and Whitney's Theorem, we omit the proof.

LEMMA 2.8. *Let $f \in \mathbf{C}[a, b]$, and let p be the quadratic polynomial interpolating f at a, $(a+b)/2$ and b. Then*

$$\|f - p\|\ [a, b] \leqslant C_0 \omega_3(f, b-a), \tag{2.16}$$

with $C_0 > 0$ an absolute constant.

3. PROOF OF MAIN THEOREM

We are now in a position to prove our main theorem.

Proof of Theorem 1.1. The proof will be given in three parts: construction of overlapping quadratics, an algorithm that transits from each quadratic to the one next to it, and error analysis. Given n, the final goal is a $\mathbf{C}^1$ convex quadratic spline s with at most kn knots in $(0, 1)$ which approximates f with an order $\omega_3(f, 1/n)$. We mentioned the positive integer k at the beginning of this paper, whose value will be determined as 8 later in the proof. Denote $\varepsilon := C_0 \omega_3(f, 1/n)$ with C_0 as in Lemma 2.8, we can assume $\varepsilon > 0$, for $\varepsilon = 0$ means f is a quadratic polynomial [11, Thm. 2.59], we can simply take $s = f$.

We first construct $2n-1$ overlapping quadratics as follows: Let $x_i := i/2n$, $i = 0, \ldots, 2n$, and $I_i := [x_i, x_{i+2}]$, $i = 0, \ldots, 2n-2$. Define on each

I_i a quadratic p_i by interpolating f at x_i, x_{i+1} and x_{i+2}. Since f is convex, so is each p_i. The approximation error of p_i is given by Lemma 2.8:

$$\|f-p_i\|\ (I_i)\leqslant \varepsilon. \tag{3.1}$$

Next we give an algorithm to transit from p_{i-1} to p_i for each $i=1, 2, ..., 2n-2$, with the aid of the previous lemmas. The algorithm proceeds from the left to the right. When a transition involves a line, it may work on two pairs (three pieces) at a time, because such a transition will go half way out of $[x_i, x_{i+1}]$ and might interfere with neighboring transitions otherwise. Since f is (uniformly) continuous on $[0, 1]$, there exists a $\delta_1>0$ such that for any $x, y\in[0, 1]$, $|x-y|\leqslant\delta_1$, $|f(x)-f(y)|\leqslant\varepsilon$. Define $\delta:=\min(\delta_1, 1/4n)$, it will be used to determine the lengths of some transiting intervals J_i.

The algorithm works as a loop, in each pass of the loop, it calls one of the following "subroutines" according to the transition type indicated by the abbreviation before them. For example, LL means p_{i-1} and p_i are both lines, and LPP means p_{i-1} is a line, p_i and p_{i+1} are both parabolas, etc. The index i runs over the interval $[1, 2n-2]$, increasing by 2 in LPP or LPL, by 1 in other cases. For convenience, we consider p_{2n-1} as a parabola when choosing the subroutine, although it is not defined. We also define $t_{0,2}:=x_0$ and $t_{2n-1,1}:=x_{2n}$ for the same reason. The reader is urged to draw his/her own pictures to show the geometric nature of the arguments.

PP. This is the simplest one. We transit from p_{i-1} to p_i by a $\mathbf{C}^1$ convex spline s_i whose existence is guaranteed by Lemma 2.7. This s_i is defined on some transiting interval $J_i:=[t_{i,1}, t_{i,2}]\subset(x_i, x_{i+1})$ with a simple knot $\xi_i\in(t_{i,1}, t_{i,2})$.

LPP. Lemma 2.6 says the transiting interval $J_i=[t_{i,1}, t_{i,2}]$ for p_{i-1} and p_i will get into the interval (x_{i+1}, x_{i+2}) and therefore might interfere with $J_{i+1}=[t_{i+1,1}, t_{i+1,2}]$ unless we know the position of $t_{i+1,1}$ in advance. If $i=2n-2$ this gives no problem at all since p_{i+1}exists only in our mind, and $t_{i+1,1}=t_{2n-1,1}$ was defined as x_{2n}. If $i<2n-2$, we have to make the transition from p_i to p_{i+1} first by calling the subroutine PP, which gives the position of $J_{i+1}\subset(x_{i+1}, x_{i+2})$. We now make the transition from p_{i-1} to p_i as follows: Let $t_{i,1}:=x_{i+1}-\delta$, and $t_{i,2}:=\min(x_{i+1}+\delta, t_{i+1,1})$, ant let l_i be the chord between $(t_{i,1}, p_{i-1}(t_{i,1}))$ and $(t_{i,2}, p_i(t_{i,2}))$. The difference $\phi_i:=p_i-l_i$ is a convex parabola with

$$\phi_i(t_{i,1})=p_i(t_{i,1})-l_i(t_{i,1})=p_i(t_{i,1})-p_{i-1}(t_{i,1}),$$

$$\phi_i(t_{i,2})=p_i(t_{i,2})-l_i(t_{i,2})=0.$$

Since p_{i-1} is the chord of p_i on $[x_i, x_{i+1}]$ which contains $t_{i,1}$, we have $\phi_i(t_{i,1})<0$. Therefore there is another point x' such that $\phi_i(x')=0$, i.e., $l_i(x')=p_i(x')$, with $x'<t_{i,1}<x_{i+1}<t_{i,2}$ since $\lim_{x\to-\infty}\phi_i(x)=+\infty$. The situation here satisfies the conditions of Lemma 2.6, hence we can find on J_i a convex quadratic spline s_i transiting from p_{i-1} to p_i in the sense of (1.4), with

$$l_i(x)\geq s_i(x)\geq \max(p_{i-1}(x), p_i(x)), \qquad x\in J_i \tag{3.2}$$

LPL. That p_{i-1} and p_{i+1} are lines implies the restrictions of f on I_{i-1} and I_{i+1} are line segments, with $f\equiv p_{i-1}$ on I_{i-1} and $f\equiv p_{i+1}$ on I_{i+1}, we are actually smoothing f itself. We abandon p_i, and transit from p_{i-1} to p_{i+1} near x_{i+1} as follows: Let $t_{i,1}:=x_{i+1}-\delta$, $t_{i+1,2}:=x_{i+1}+\delta$, and l_i be the chord of f between $t_{i,1}$ and $t_{i+1,2}$. Since subtracting the equation of l_i from those of l_i, p_{i-1} and p_{i+1} will result in an isosceles triangle with the base l_i horizontal, it is clear that we can apply Lemma 2.2 (through a linear change of variable) to transit from p_{i-1} to p_{i+1} by a convex quadratic polynomial on $[t_{i,1}, t_{i+1,2}]$. It is trivial to see from the properties of convex functions that this polynomial lies within the triangle enclosed by l_i, p_{i-1}, and p_{i+1}. The subscripts of $t_{i+1,2}$ are not what one may expect, the reason is that in this proof by $J_i=[t_{i,1}, t_{i,2}]$ and s_i we always mean the transiting interval and parabola (or spline) for p_{i-1} and p_i, but here we abandoned p_i and are working on p_{i-1} and p_{i+1}. To further keep our notation consistent, we define $t_{i,2}:=t_{i+1,1}:=x_{i+1}$, and split the transiting interval and parabola into two pieces at this point: let $J_i:=[t_{i,1}, t_{i,2}]$, $J_{i+1}:=[t_{i+1,1}, t_{i+1,2}]$, define s_i as the part of the transiting parabola on J_i, and s_{i+1} as the part on J_{i+1}. This will be convenient later in the definition (3.3) of the final approximating spline.

LL. That p_{i-1} and p_i are both lines implies the restriction of f on $[x_{i-1}, x_{i+2}]$ is a line segment, and f, p_{i-1} and p_i are all identical. Although we have nothing to do here, we still like to define $J_i:=[t_{i,1}, t_{i,2}]:=[x_i, x_i]$ as a one-point-set and $s_i:=f$ on J_i for the same reason as in LPL.

PL. We note that p_{i-2} can not be a line if this subroutine is called, since that would be an LPL case in the last pass of the loop and have already been done. So we only have two cases here: p_{i-2} is a parabola or $i=1$. In either case $t_{i-1,2}$ has been defined and is less than x_i, therefore we can transit from p_{i-1} to p_i as follows: Define $t_{i,1}:=\max(x_i-\delta, t_{i-1,2})$ and $t_{i,2}:=x_i+\delta$. Let l_i be the chord between $(t_{i,1}, p_{i-1}(t_{i,1}))$ and $(t_{i,2}, p_i(t_{i,2}))$. An argument very similar to that in LPP shows that we can find an s_i on $J_i:=[t_{i,1}, t_{i,2}]$ with the same properties.

We are now ready to define the final approximating spline s. Since s_i

always denotes the spline piece transiting from p_{i-1} to p_i on $J_i = [t_{i,1}, t_{i,2}]$, it is natural to define

$$s(x) := \begin{cases} s_i(x), & x \in J_i, \quad i = 1, ..., 2n-2 \\ p_i(x), & t_{i,2} < x < t_{i+1,1}, \quad i = 0, ..., 2n-2. \end{cases} \tag{3.3}$$

It is obvious from the construction that s is a $\mathbf{C}^1$ quadratic spline. It is also obvious that s is convex on $[0, 1]$ since it is piecewise convex and has a continuous first derivative. As for the number of its interior knots, there are $2n-1$ x_i's in $(0, 1)$; for the transition of each of the $2n-2$ pairs of overlapping p_i's, no more than three knots were added, this gives no more than $6n-6$ additional knots, therefore s has no more than $8n-7 < 8n$ interior knots, i.e., we can take $k = 8$ as claimed at the beginning of the proof.

The last part of the proof is error analysis. If $t_{i,2} < x < t_{i+1,1}$ for some i, by (3.3) and (3.1) we have

$$|f(x) - s(x)| = |f(x) - p_i(x)| \leqslant \varepsilon. \tag{3.4}$$

If $x \in J_i$ for some i, we estimate the error according to the transition type. In type PP, s_i lies between p_{i-1} and p_i, (3.4) still holds by (3.1). In type LL, all functions are identical, there is no error at all. The remaining three types are basically the same, the following estimate is for PL and (the LP part of) LPP types. For LPL, only the notation needs changes, because we divided the transiting part and interval into two parts. We recall that in PL and LPP, s_i transits from p_{i-1} to p_i on $J_i := [t_{i,1}, t_{i,2}]$, with $t_{i,1} < \bar{x} < t_{i,2}$, $|t_{i,1} - \bar{x}| \leqslant \delta$ and $|t_{i,2} - \bar{x}| \leqslant \delta$, where $\bar{x} = x_i$ for PL and $\bar{x} = x_{i+1}$ for LPP, thus by the choice of δ

$$|f(x) - f(\bar{x})| \leqslant \varepsilon, \qquad \text{any} \quad x \in J_i. \tag{3.5}$$

We also recall that l_i is the chord between the points $(t_{i,1}, p_{i-1}(t_{i,1}))$ and $(t_{i,2}, p_i(t_{i,2})$, therefore by (3.1) and (3.5)

$$|l_i(t_{i,2}) - f(\bar{x})| = |p_i(t_{i,2}) - f(\bar{x})| \leqslant |p_i(t_{i,2}) - f(t_{i,2})| + |f(t_{i,2}) - f(\bar{x})| \leqslant 2\varepsilon.$$

Similarly,

$$|l_i(t_{i,1}) - f(\bar{x})| \leqslant 2\varepsilon.$$

Since l_i is a line segment, we have for any $x \in J_i$

$$|l_i(x) - f(\bar{x})| \leqslant 2\varepsilon,$$
$$|l_i(x) - f(x)| \leqslant |l_i(x) - f(\bar{x})| + |f(\bar{x}) - f(x)| \leqslant 2\varepsilon + \varepsilon = 3\varepsilon.$$

Combine this with (3.1) and (3.2) we obtain

$$|s_i(x) - f(x)| \leqslant 3\varepsilon, \qquad \text{for any} \quad x \in J_i.$$

We now finish this proof by stating

$$|s(x) - f(x)| \leqslant 3\varepsilon = 3C_0\omega_3(f, 1/n), \qquad x \in [0, 1],$$

or

$$\|f - s\| \leqslant C\omega_3(f, 1/n)$$

with $C := 3C_0$ an absolute constant. ∎

ACKNOWLEDGMENTS

The author thanks Professor Ronald A. DeVore for interesting him in this problem and spending time to help him improve the manuscript, and thanks Professor Xiang-Min Yu for helpful talks and suggestions at several stages of this work. Thanks also go to Professor Dany Leviatan and the referees of this paper for their suggestions on improving the manuscript.

REFERENCES

1. R. K. BEATSON, Convex approximation by splines, *SIAM J. Math. Anal.* **12** (1981), 549–559.
2. R. K. BEATSON, Monotone and convex approximation by splines: Error estimates and a curve fitting algorithm, *SIAM J. Numer. Anal.* **19** (1982), 1278–1285.
3. C. K. CHUI, P. W. SMITH, AND J. D. WARD, Degree of $\mathbf{L}_p$ approximation by monotone splines, *SIAM J. Math. Anal.* **11** (1980), 436–447.
4. R. A. DEVORE, Monotone approximation by splines, *SIAM J. Math. Anal.* **8** (1977), 891–905.
5. R. A. DEVORE AND Z. YAN, Error analysis for piecewise quadratic curve fitting algorithms, *Comput. Aided Geom. Design* **3** (1986), 205–215.
6. Y. HU, Convexity preserving approximation by free knot splines, *SIAM J. Math. Anal.* **22** (1991), 1183–1191.
7. D. LEVIATAN AND H. N. MHASKAR, The rate of monotone spline approximation in the $\mathbf{L}_p$-norm, *SIAM J. Math. Anal.* **13** (1982), 866–874.
8. D. F. MCALLISTER, E. PASSOW, AND J. A. ROULIER, Algorithms for computing shape preserving spline interpolations to data, *Math. Comp.* **31** (1977), 717–725.
9. D. F. MCALLISTER AND J. A. ROULIER, Interpolation by convex quadratic splines, *Math. Comp.* **32** (1978), 1154–1162.
10. H. L. ROYDEN, "Real Analysis," 2nd ed., Macmillan, New York, 1968.
11. L. L. SCHUMAKER, "Spline Functions: Basic Theory," Wiley, New York, 1981.
12. L. L. SCHUMAKER, On shape preserving quadratic spline interpolation, *SIAM J. Numer. Anal.* **20** (1983), 294–325.
13. X. WANG, On DeVore's conjecture, *J. Zhejing Univ.* **20** (1986), 106–108.